A surjectively universal Polish group which is not universal

Ruiwen Li (joint with Longyun Ding and Bo Peng)

Nankai University

Descriptive Dynamics and Combinatorics Seminar
October 7, 2025

Preliminary

A Polish group is a topological group whose topology is a Polish topology. For a Polish group G and Polish space X, a Borel group action is a group action a of G on X, and as a function from $G \times X$ to X it is Borel. The orbit equivalence relation E_G^X induced by a is defined as:

$$xE_G^X y \Leftrightarrow \exists g \in G, a(g,x) = y.$$

The space X together with the action a is also called a Borel G-space.

For an equivalence relation E defined on a Polish space X and another equivalence relation F defined on a Polish space Y, if there is a Borel map f from X to Y such that for any $x, y \in X$, we have

$$xEy \Leftrightarrow f(x)Ff(y)$$

then we say E is Borel reducible to F, denoted by $E \leq_B F$.

Theorem(Gao–Jackon)
Let G be a countable abelian group and E_G^X be induced by a Borel G-action, then E_G^X is hyperfinite.

Theorem(Mackey–Hjorth)
Let G be a Polish group and H be a closed subgroup of it. Then for every equivalence relation E_H^X induced by a Borel H-action, there is a Borel G-action on some Polish space Y such that E_H^X is Borel reducible to E_G^Y.

Theorem(Mackey–Hjorth)
Let G be a Polish group and H be a closed subgroup of it. Then for every equivalence relation E_H^X induced by a Borel H-action, there is a Borel G-action on some Polish space Y such that E_H^X is Borel reducible to E_G^Y.

Theorem(Becker–Kechris)
For every Polish group G, there is a universal Borel G-space.

We denote the induced equivalence relation on the universal Borel G-space by E_G.

Corollary

Let G be a Polish group and H be a closed subgroup of it, then $E_H \leq_B E_G$.

Corollary

Let G be a Polish group and H be a closed subgroup of it, then $E_H \leq_B E_G$.

Definition

A Polish group G is universal if every Polish group is isomorphic to a closed subgroup of G.

Corollary

Let G be a Polish group and H be a closed subgroup of it, then $E_H \leq_B E_G$.

Definition

A Polish group G is universal if every Polish group is isomorphic to a closed subgroup of G.

Corollary

If a given Polish group G is universal, then E_G is a complete orbit equivalence relation.

Theorem(Zielinski)
The homeomorphism relation on compact Polish spaces is a complete orbit equivalence relation.

Theorem(Gao-Kechris, Clemens)
The isometry relation of Polish metric spaces is a complete orbit equivalence relation.

Open Problem(Sabok)

If E_G is a complete orbit equivalence relation, is G necessarily a universal Polish group?

Fact

Let G be a Polish group and H be a quotient Polish group of it. Then for every equivalence relation E_H^X induced by a Borel H-action, it can also be induced by a Borel G-action X.

Definition

A surjectively universal Polish group G is such that every Polish group is isomorphic to a topological quotient group of G.

Definition

A surjectively universal Polish group G is such that every Polish group is isomorphic to a topological quotient group of G.

Corollary

If a given Polish group G is surjectively universal, then E_G is a complete orbit equivalence relation.

Theorem(Ding)
There exists a surjectively universal Polish group.

Theorem(Ding)
There exists a surjectively universal Polish group.

Theorem(Ding-L.-Peng)
There exists a surjectively universal Polish group that is not universal.

The Graev metric group

Let X be a set, then take $e \neq X$ and $X^{-1} = \{x^{-1} : x \in X\}$. Denote $X \cup X^{-1} \cup \{e\}$ by $\bar{X}$. Let $e^{-1} = e$ and $(x^{-1})^{-1} = x$.

Let X be a set, then take $e \neq X$ and $X^{-1} = \{x^{-1} : x \in X\}$. Denote $X \cup X^{-1} \cup \{e\}$ by $\bar{X}$. Let $e^{-1} = e$ and $(x^{-1})^{-1} = x$.

We denote the set of words over $\bar{X}$ by $W(X)$, in other words, $W(X) = \bar{X}^{<\omega}$

For w in $W(X)$, if e and xx^{-1} don't occur in w as a subword for every $x \in \bar{X}$, then we say w is irreducible. The collection of irreducible words is denoted by $F(X)$. We use the symbol $\Box$ to represent the empty word.

For w in $W(X)$, if e and xx^{-1} don't occur in w as a subword for every $x \in \bar{X}$, then we say w is irreducible. The collection of irreducible words is denoted by $F(X)$. We use the symbol $\Box$ to represent the empty word.

For w in $W(X)$, by deleting its subwords with the form e or xx^{-1} where $x \in \bar{X}$, we get a irreducible word, denoted by w'. we say w is a trivial extension of w'.

For w in $W(X)$, if e and xx^{-1} don't occur in w as a subword for every $x \in \bar{X}$, then we say w is irreducible. The collection of irreducible words is denoted by $F(X)$. We use the symbol $\Box$ to represent the empty word.

For w in $W(X)$, by deleting its subwords with the form e or xx^{-1} where $x \in \bar{X}$, we get a irreducible word, denoted by w'. we say w is a trivial extension of w'.

The set $F(X)$ is a group under the operation $u \cdot v = (u \frown v)'$, $\Box$ is the neutral element.

If X is a metric space with metric d, extend d to $\bar{X}$ such that for every $x, y \in X$,

$$d(x^{-1}, y^{-1}) = d(x, y),$$

$$d(x, y^{-1}) = d(x, e) = d(x^{-1}, e) = 1.$$

For $u = x_0 \cdots x_n$ and $v = y_0 \cdots y_n$ in $W(X)$, let

$$\rho(u, v) = \Sigma_{0 \leq i \leq n} d(x_i, y_i).$$

And for $u, v \in F(X)$, let

$$d(u, v) = \inf\{\rho(u^*, v^*) : (u^*)' = u, (v^*)' = v, lh(u) = lh(v)\}.$$

Let $m, n \in \mathbb{N}$ and $m \leq n$. A bijection θ on $\{m, \cdots, n\}$ is a match if

(1) $\theta^2 = \mathrm{id}$;

(2) there is no $i, j \in \{m, \cdots, n\}$ such that $i < j < \theta(i) < \theta(j)$.

Given $w = x_0 \cdots x_n \in W(X)$ and a match θ on $\{0, \cdots, n\}$, let

$$x_i^\theta = \begin{cases} x_i, & \theta(i) > i. \\ e, & \theta(i) = i. \\ x_{\theta(i)}^{-1}, & \theta(i) < i. \end{cases}$$

and $w^\theta = x_0^\theta \cdots x_n^\theta$.

Given $w = x_0 \cdots x_n \in W(X)$ and a match θ on $\{0, \cdots, n\}$, let

$$x_i^\theta = \begin{cases} x_i, & \theta(i) > i. \\ e, & \theta(i) = i. \\ x_{\theta(i)}^{-1}, & \theta(i) < i. \end{cases}$$

and $w^\theta = x_0^\theta \cdots x_n^\theta$.

Theorem Given $w = x_0 \cdots x_n \in W(X)$ and a match θ on $\{0, \cdots, n\}$, we have that $(w^\theta)' = \Box$.

Theorem(Ding-Gao)
For every $u \in F(X)$, $d(u, \Box) = \min\{\rho(u, u^\theta) : \theta \textit{ is a match}\}$.

Theorem(Ding-Gao)
For every $u \in F(X)$, $d(u, \Box) = \min\{\rho(u, u^\theta) : \theta \textit{ is a match}\}$.

Theorem(Graev)
Let (X, d) be a metric space, then the d can be extended to a two-sided invariant metric on $F(X)$. Furthermore, $F(X)$ is a topological group in the topology induced by d.

Let $(\bar{F}(X), d)$ be the completion of $(F(X), d)$, then $(\bar{F}(X), d)$ is a Polish group.

On the Baire space ω^ω we have the canonical metric d such that for $x \neq y \in \omega^\omega$, $d(x, y) = 2^{-n}$, where n is the least number such that $x(n) \neq y(n)$. The group $(\bar{F}(\omega^\omega), d)$ is called the Graev metric group.

Let $(\bar{F}(X), d)$ be the completion of $(F(X), d)$, then $(\bar{F}(X), d)$ is a Polish group.

On the Baire space ω^ω we have the canonical metric d such that for $x \neq y \in \omega^\omega$, $d(x, y) = 2^{-n}$, where n is the least number such that $x(n) \neq y(n)$. The group $(\bar{F}(\omega^\omega), d)$ is called the Graev metric group.

Theorem

The Graev metric group is a surjectively universal tsi Polish group.

New metrics on free groups

Let $\mathbb{R}_+$ denote the set of non-negative real numbers. A function $\Gamma : \bar{X} \times \mathbb{R}_+ \to \mathbb{R}_+$ is a scale on $\bar{X}$ if the following hold for any $x \in \bar{X}$ and $r \in \mathbb{R}_+$

(i) $\Gamma(e, r) = r$, $\Gamma(x, r) \geq r$;

(ii) $\Gamma(x, r) = 0$ iff $r = 0$;

(iii) $\Gamma(x, \cdot)$ is a monotone increasing function with respect to the second variable;

(iv) $\lim_{r \to 0} \Gamma(x, r) = 0$.

Example

Take $\Gamma(x, r) = r$ for every $x \in \bar{X}$, then Γ is the trivial scale.

Example

Take $\Gamma(x, r) = r$ for every $x \in \bar{X}$, then Γ is the trivial scale.

Example

Let G be a metrizable group and d_G be a compatible left-invariant metric on G. Define $\Gamma_G : G \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\Gamma_G(g, r) = \max\{r, \sup\{d_G(1_G, g^{-1}hg) : d_G(1_G, h) \leq r\}\}.$$

It is easy to see that Γ_G satisfies the conditions (i)–(iv) in the definition of a scale. We will also call Γ_G the scale on G.

Let Γ be a scale on X. For $l \in \mathbb{N}$, $w \in W(X)$ with $lh(w) = l + 1$ and θ a match on $0, \cdots, l$, define $N_\Gamma^\theta(w)$ by induction on l as follows:

(0) for $l = 0$, let $w = x$ and define $N_\Gamma^\theta(w) = d(e, x)$;

(1) if $l > 0$ and $\theta(0) = k < l$, let $\theta_1 = \theta \restriction 0, \cdots, k$, $\theta_2 = \theta \restriction k+1, \cdots, l$ and $w = w_1 \frown w_2$ where $lh(w1) = k + 1$; define

$$N_\Gamma^\theta(w) = N_\Gamma^{\theta_1}(w_1) + N_\Gamma^{\theta_2}(w_2);$$

(2) if $l > 0$ and $\theta(0) = l$, let $\theta_1 = \theta \restriction 1, \cdots, l-1$ and $w = x^{-1} w_1 y$ where $x, y \in \bar{X}$; then $lh(w_1) = l - 1$. Define

$$N_\Gamma^\theta(w) = d(x, y) + \max\{\Gamma(x, N_\Gamma^{\theta_1}(w_1)), \Gamma(y, N_\Gamma^{\theta_1}(w_1))\}.$$

For $w \in F(X)$, define

$$N_\Gamma(w) = \inf\{N_\Gamma^\theta(w^*) : (w^*)' = w, \theta \textit{ is a match}\}.$$

And $d_\Gamma(u, v) = N_\Gamma(u^{-1}v)$ for $u, v \in F(X)$.

For $w \in F(X)$, define

$$N_\Gamma(w) = \inf\{N_\Gamma^\theta(w^*) : (w^*)' = w, \theta \textit{ is a match}\}.$$

And $d_\Gamma(u, v) = N_\Gamma(u^{-1}v)$ for $u, v \in F(X)$.

Fact

For every scale Γ and $u, v \in F(X)$, $d_\Gamma(u, v) \geq d(u, v)$ where d is the Graev metric.

Theorem(Ding-Gao)
Let (X, d) be a metric space, Γ is a scale on $\bar{X}$, then the d_Γ is a left invariant metric on $F(X)$ extending d. Furthermore, $F(X)$ is a topological group in the topology induced by d_Γ, denote it by $F_\Gamma(X)$.

Theorem(Ding-Gao)
Let (X, d) be a metric space, Γ is a scale on $\bar{X}$, then the d_Γ is a left invariant metric on $F(X)$ extending d. Furthermore, $F(X)$ is a topological group in the topology induced by d_Γ, denote it by $F_\Gamma(X)$.

Let $d_\Gamma^{-1}(u, v) = d_\Gamma(u^{-1}, v^{-1})$, and $\Delta_\Gamma = d_\Gamma^{-1} + d_\Gamma$, then Δ_Γ is a compatible metric on topological group $F_\Gamma(X)$. The completion of $(F_\Gamma(X), \Delta_\Gamma)$ is a Polish group, denoted by $\bar{F}_\Gamma(X)$.

Theorem(Ding-Gao)

Let G be a topological group and d_G a compatible left-invariant metric on G. Let Γ be a scale on $\overline{X}$. Let $\varphi : \bar{X} \to G$ be a function. Suppose that for any $x, y \in \overline{X}$ and $r \in \mathbb{R}_+$:

(a) $\varphi(e) = 1_G$; $\varphi(x^{-1}) = \varphi(x)^{-1}$;

(b) $d_G(\varphi(x), \varphi(y)) \leq d(x, y)$; and

(c) $\Gamma_G(\varphi(x), r) \leq \Gamma(x, r)$.

Then φ can be uniquely extended to a continuous group homomorphism $\Phi : F(X) \to G$ such that for any $w \in F(X)$

$$d_G(\Phi(w), 1_G) \leq N_\Gamma(w)$$

Theorem(Ding)
There is a scale on $\overline{\omega^\omega}$ such that $\bar{F}_\Gamma(\omega^\omega)$ is a surjectively universal Polish group.

Main theorem

Theorem(Ding-L.-Peng)
There exists a surjectively universal Polish group that is not universal. In particular, there is a non-universal Polish group that induces a complete orbit equivalence relation.

Let

$$\mathcal{N}_n = \{x \in \omega^\omega : \forall m \geq n, x(m) = 0\}.$$

and for $x \in \omega^\omega$, let

$$\pi_n(x)(m) = \begin{cases} x(m), & m < n. \\ 0, & m \geq n. \end{cases}$$

Then for $u, v \in F(\mathcal{N}_n)$, if $u \neq v$, we have $d_\Gamma(u, v) \geq 2^{-n}$, so $F(\mathcal{N}_n)$ is a discrete closed subgroup of $\overline{F}_\Gamma(\omega^\omega)$, denote it by $F_\Gamma(\mathcal{N}_n)$.

Step 1

The map π_n can be uniquely extended to a continuous homomorphism f_n from $\overline{F}_\Gamma(\omega^\omega)$ to $F_\Gamma(\mathcal{N}_n)$ by the theorem we mentioned.

Step 1

The map π_n can be uniquely extended to a continuous homomorphism f_n from $\overline{F}_\Gamma(\omega^\omega)$ to $F_\Gamma(\mathcal{N}_n)$ by the theorem we mentioned.

Theorem(Ding-Gao)

Let G be a topological group and d_G a compatible left-invariant metric on G. Let Γ be a scale on $\overline{X}$. Let $\varphi : X \to G$ be a function. Suppose that for any $x, y \in \overline{X}$ and $r \in \mathbb{R}_+$:

(a) $\varphi(e) = 1_G$; $\varphi(x^{-1}) = \varphi(x)^{-1}$;

(b) $d_G(\varphi(x), \varphi(y)) \leq d(x, y)$; and

(c) $\Gamma_G(\varphi(x), r) \leq \Gamma(x, r)$.

Then φ can be uniquely extended to a continuous group homomorphism $\Phi : F(X) \to G$ such that for any $w \in F(X)$

$$d_G(\Phi(w), 1_G) \leq N_\Gamma(w)$$

Step 2

For $g \neq h \in \overline{F}_\Gamma(\omega^\omega)$, there is some $n \in \mathbb{N}$ such that $f_n(g) \neq f_n(h)$. So the homomorphism from $\overline{F}_\Gamma(\omega^\omega)$ to $\prod_{n\in\mathbb{N}} F(\mathcal{N}_n)$ is continuous and injective.

Step 3

The group $(\mathbb{R}, +)$ is connected, it cannot be continuously embedded to totally disconnected group $\prod_{n\in\mathbb{N}} F(\mathcal{N}_n)$. Moreover, by the automatic continuity of $\mathrm{Iso}(\mathbb{U})$, $\mathrm{Iso}(\mathbb{U})$ even cannot be an abstract subgroup of $\prod_{n\in\mathbb{N}} F(\mathcal{N}_n)$.

Step 3

The group $(\mathbb{R}, +)$ is connected, it cannot be continuously embedded to totally disconnected group $\prod_{n\in\mathbb{N}} F(\mathcal{N}_n)$. Moreover, by the automatic continuity of $\mathrm{Iso}(\mathbb{U})$, $\mathrm{Iso}(\mathbb{U})$ even cannot be an abstract subgroup of $\prod_{n\in\mathbb{N}} F(\mathcal{N}_n)$.

Theorem(Sabok)

For every separable topological group H and an abstract homomorphism from $\mathrm{Iso}(\mathbb{U})$ to H, the homomorphism is continuous.

Thank You!